More about the nonstandard R-matrix associated with $SU_q(2)$

1993 J. Phys. A: Math. Gen. 26 443

(http://iopscience.iop.org/0305-4470/26/2/027)

J. Phys. A: Math. Gen. 26 (1993) 443-447. Printed in the UK

COMMENT

More about the non-standard R-matrix associated with $SU_q(2)$

Mo-Lin Ge and Yi-Wen Wang

Theoretical Physics Section, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China

Received 23 September 1992

Abstract. The Markov trace is constructed for the general R-matrices associated with $SU_q(2)$ at q a root of unity. The relationship between the R-matrices derived based on quantum algebra and those obtained by Akutsu *et al* is clarified.

1. Introduction

The 'non-standard' family of R-matrices associated with the spin model was firstly discussed by Lee *et al* [1] and can simply be interpreted in terms of the representations of quantum algebra [2–4]. In [2–4] we presented general formulae of coloured R-matrices on the basis of a direct application of Drinfeld preliminary theory [5] at q a root of unity. The main calculations were made through the q-boson realization of quantum algebra for $SL_q(2)$ [6, 3].

It is known that for the q-commutators

$$[J_+, J_-] = [J_0] \qquad [J_0, J_\pm] = \pm 2J_\pm \tag{1.1}$$

where

$$[n] = (q^n - q^{-n})/(q - q^{-1}) \tag{1.2}$$

the mapping $SL_q(2) \to B_q$ defines a q-boson realization of $SL_q(2)$ with generators $J_\pm$ and J_0

$$\begin{aligned} J_+ &\to B(J_+) = a^+\alpha(N) \\ J_- &\to B(J_-) = a\beta(N) \qquad (N = a^+a) \\ J_0 &\to B(J_0) = 2N - \lambda \end{aligned} \tag{1.3}$$

with

$$aa^+ - q^{-1}a^+a = q^N \qquad [N, a^+] = a^+ \qquad [N, a] = -a$$

if $\alpha(N)$ and $\beta(N)$ satisfy

$$\alpha(N-1)\beta(N) = [\lambda + 1 - N] \qquad \lambda \in \mathbb{C}. \tag{1.4}$$

After calculations we find that the formulae of Drinfeld give [2–4]

$$\begin{aligned} R(\lambda_1, \lambda_2) = q^{(2N-\lambda_1)\otimes(2N-\lambda_2)/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} q^{n(n-1)/2} \{q^{N-\lambda_1/2} a^+ \alpha(N, \lambda_1) \\ \otimes q^{-N+\lambda_2/2} a\beta(N, \lambda_2)\}^n \end{aligned} \tag{1.5}$$

where

$$\alpha(N-1,\lambda_i)\beta(N,\lambda_i)=[\lambda_i+1-N] \qquad i=1,2. \tag{1.6}$$

Equation (1.5) leads to the universal R-matrix form [2-4]

$$(\check{R}^{j_1j_2}(\lambda,\mu))^{ab}_{mi}$$

$$=q^{2(j_1+b-\frac{1}{2}\lambda)(j_2+a-\frac{1}{2}\mu)}\left\{\delta^a_i\delta^b_m+\sum_{n=0}^{k}\frac{(1-q^{-2})^n}{[n]!}q^{-\frac{1}{2}n(n-1)+n(j_1-j_2+b-a-\frac{1}{2}\lambda+\frac{1}{2}\mu)}\right.$$

$$\left.\times\prod_{L=0}^{n}\alpha_{j_1,m+L-1}(\lambda)\beta_{j_2,i-L+1}(\mu)[j_2+i-L+1]\delta^b_{m+n}\delta^a_{i-n}\right\} \tag{1.7}$$

where $k=\min(2j_1,2j_2)$, $\alpha_{j,m}(\mu)=\alpha(j+m,\mu)$, $\beta_{j,m}(\mu)=\beta(j+m,\mu)$ and $q^{2k+1}=1$. It is noted that q is a cyclic parameter whereas λ and μ are continuous ones and are referred to as 'colours'. Equation (1.7) satisfies

$$\check{R}_{12}(\lambda,\mu)\check{R}_{23}(\lambda,\nu)\check{R}_{12}(\mu,\nu)=\check{R}_{23}(\mu,\nu)\check{R}_{12}(\lambda,\nu)\check{R}_{23}(\lambda,\mu).$$

All of the known coloured or non-coloured non-standard R-matrices associated with SU(2) are special case of (1.7) [4].

In this note we shall firstly calculate the inverse of $\check{R}^{j_1j_2}(\lambda,\mu)$ and construct the Markov trace. Secondly we shall prove that by making the coloured symmetry breaking transformation, (1.7) leads to all the results of [7-9].

2. Markov trace properties

Equation (1.7) can be recast into the form

$$(\check{R}^{j_1j_2}(\lambda,\mu))^{ab}_{mi}=q^{2(j_1+b-\frac{1}{2}\lambda)(j_2+a-\frac{1}{2}\mu)}\left\{\delta^a_i\delta^b_m+\sum_{n=1}^{k}A(n;a,b,m,i)\delta^a_{i-n}\delta^b_{m+n}\right\} \tag{2.1}$$

where

$$A(n;a,b,m,i)=\frac{(q-q^{-1})^n}{[n]!}q^{\frac{1}{2}[-n(n+1)]+n(j_1-j_2-\frac{1}{2}\lambda+\frac{1}{2}\mu+b-a)}C(n;m,i) \tag{2.2}$$

$$C(n;m,i)=\prod_{L=1}^{n}\alpha_{j_1,m+L-1}(\lambda)B_{j_2,i-L+1}(\mu)[j_2+i-L+1]. \tag{2.3}$$

To satisfy

$$\check{R}^{j_1j_2}(\lambda,\mu)\bar{\check{R}}^{j_1j_2}(\lambda,\mu)=I \tag{2.4}$$

we substitute the form

$$(\bar{\check{R}}^{j_1j_2}(\lambda,\mu))^{mi}_{cd}=q^{-\Delta}\left\{\delta^m_d\delta^i_c+\sum_{n=1}B(n;c,d,m,i)\delta^m_{d+n}\delta^i_{c-n}\right\} \tag{2.5}$$

into (2.4) to determine $B(n;c,d,m,i)$ where

$$\Delta=2(j_1+m-\tfrac{1}{2}\lambda)(j_2+i-\tfrac{1}{2}\mu). \tag{2.6}$$

After calculations we find that

$$(\bar{\check{R}}^{j_1 j_2}(\lambda, \mu))^{ab}_{cd} = q^{-\Delta}\left\{\delta^a_d \delta^b_c + \sum_{n=1}^{k} \frac{(q^{-1}-q)^n}{[n]!} q^{\frac{1}{2}[n(n+1)]+n(j_2-j_1+\frac{1}{2}\lambda-\frac{1}{2}\mu+b-a)} C(n; b, a)\delta^a_{d+n}\delta^b_{c-n}\right\} \tag{2.7}$$

where $C(n; i, m)$ are given by (2.3) with $m \leftrightarrow i$. In the derivation of (2.7) the relation

$$\prod_{m=0}^{n-1} (1+q^{-2m}Z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(1-n)} Z^m \tag{2.7a}$$

when

$$Z=-1 \qquad \sum_{m=0}^{n} \frac{(-1)^m [n]!}{[m]![n-m]!} q^{m(1-n)} = 0$$

has been used (for the definition of $[{n \atop m}]_q$, see (3.2)). When $\lambda = \mu$ and $j_1 = j_2$ (2.7) turns out to be the known results given by [1].

In order to discuss the Markov trace [7, 10] and Alexander polynomials [1, 11] only the case $\lambda = \mu$ should be considered and a diagonal matrix h (or h') should be found such that [11]

$$\mathrm{tr}_2(\check{R}(I \otimes h)) = \tau I \qquad \mathrm{tr}_2(\bar{\check{R}}(I \otimes h')) = \bar{\tau} I \tag{2.8}$$

where τ and $\bar{\tau}$ are scalars and tr_2 means that the trace is taken on the second space only. Equation (2.8) requires

$$\check{R}^{jj}_{jj} h_j = \ldots = \check{R}^{j-n,j-n}_{j-n,j-n} h_{j-n} + \sum_{i=1}^{n} \check{R}^{j-n,j-n+i}_{j-n,j-n+i} h_{j-n+i} = \ldots \tag{2.9}$$

$$\bar{\check{R}}^{-j-j}_{-j-j} h'_{-j} = \ldots = \bar{\check{R}}^{-j+n,-j+n}_{-j+n,-j+n} h'_{-j+n} + \sum_{i=1}^{n} \bar{\check{R}}^{-j+n,-j+n-i}_{-j+n,-j+n-i} h'_{-j+n-i} = \ldots. \tag{2.10}$$

Through direct calculations we obtain

$$h = h' = \mathrm{diag}(1, q^2, \ldots, q^{4j}) \qquad \mathrm{tr}\, h = 0 \tag{2.11}$$

and

$$\tau = q^{\frac{1}{2}\lambda^2 - 4j\lambda + 1} \qquad \bar{\tau} = q^{-\frac{1}{2}\lambda^2}. \tag{2.12}$$

The general proof of satisfaction of Murakami's redundant conditions [11] for the enhanced Yang–Baxter operators associated with R is difficult. The rigorous verification of the existence of Alexander-type link polynomials for $j = 1$ has been given explicitly in [12].

3. Coloured symmetry breaking transformation and the consequences

Akutsu *et al* [7–9] presented the solutions of the R-matrix

$$G^{ab}_{cd}(\alpha, \beta; +) = \begin{bmatrix} a \\ d \end{bmatrix}_\omega \begin{pmatrix} c \\ b \end{pmatrix}_{\beta,\omega} \beta^d \omega^{bd} \alpha^{ub+vc} \beta^{-ud-va} f(\alpha, \beta, \omega) \times \frac{F(\alpha, a)F(\beta, b)}{F(\alpha, d)F(\beta, c)} e^{w(a+d-b-c)+p(ab-cd)} \tag{3.1}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{cases} (q;\, m)_q/[(q;\, m-n)_q(q;\, n)_q] & \text{for } m-n \geq 0 \\ 0 & \text{for } m-n<0 \end{cases} \tag{3.2}$$

$$\begin{pmatrix} m \\ n \end{pmatrix}_{z,q} = \frac{(z;\, m)_q}{(z;\, n)_q} \qquad \text{for } m, n \geq 0 \tag{3.3}$$

$$(z;\, n) = \begin{cases} \Pi_{k=0}^{n-1}(1-zq^k) & \text{for } n>0 \\ 0 & \text{for } n=0. \end{cases} \tag{3.4}$$

We shall prove that (3.1) is the consequence of coloured symmetry breaking transformation of (2.1) with $j_1 = j_2$. The proof is based on the following statements:

(1) If $\check{R}(\alpha, \beta)$ satisfies

$$\sum_{j,k,l} \check{R}^{ab}_{jk}(\alpha, \beta)\check{R}^{kn}_{lf}(\alpha\gamma)\check{R}^{jl}_{cd}(\beta, \gamma) = \sum_{j,k,l} \check{R}^{bn}_{lj}(\beta, \gamma)\check{R}^{al}_{ck}(\alpha, \gamma)\check{R}^{kj}_{df}(\alpha, \beta) \tag{3.5}$$

then

$$\tilde{R}^{ab}_{cd}(\alpha, \beta) = (f(\alpha))^{\zeta(a,b,c,d)}(f(\beta))^{\xi(a,b,c,d)}\check{R}^{ab}_{cd}(\alpha, \beta) \tag{3.6}$$

satisfies the same equations for

$$\zeta(a, b, c, d) = ub + vc \qquad \xi(a, b, c, d) = -ud - vb \tag{3.7}$$

or

$$\zeta(a, b, c, d) = w(a+d)(c-b) \qquad \xi(a, b, c, d) = -w(a-d)(c+b). \tag{3.8}$$

(2) If $\check{R}(\alpha, \beta)$ satisfies (3.5), it does so for

$$\hat{R}^{ab}_{cd}(\alpha, \beta) = f(\alpha, \beta)\frac{F(\alpha, a)F(\beta, b)}{F(\alpha, d)F(\beta, c)}\check{R}^{ab}_{cd}(\alpha, \beta). \tag{3.9}$$

By making the transformation (3.7) and by (3.9), then (3.1) can be simplified to

$$\tilde{G}^{ab}_{cd}(\alpha, \beta; +) = \begin{bmatrix} a \\ d \end{bmatrix}_\omega \begin{pmatrix} c \\ d \end{pmatrix}_{\beta,\omega} \beta^d \omega^{bd} \tag{3.10}$$

and the inverse to

$$\tilde{G}^{ab}_{cd}(\alpha, \beta; -) = \begin{bmatrix} b \\ c \end{bmatrix}_{1/\omega} \begin{pmatrix} d \\ a \end{pmatrix}_{1/\alpha,1/\omega} \alpha^{-c}\omega^{-ac}. \tag{3.11}$$

Both satisfy (3.5). Equation (3.11) can be written as

$$\tilde{G}^{ab}_{cd}(\alpha, \beta; -) = Z^{2c}q^{2ac}\prod_{j=1}^{n}\frac{1-q^{2c+2j}}{1-q^{2j}}\prod_{k=0}^{n-1}(1-Z^2q^{2a+2k}) \qquad (n=b-c=d-a) \tag{3.12}$$

$$= q^{2ab+n(b-a)-\mu(2b-n)-\frac{1}{2}[n(n+1)]}\frac{(q-q^{-1})^n}{[n]!}\prod_{j=1}^{n}[c+j][\mu-a-j-1] \tag{3.13}$$

where $\alpha = Z^{-2}$, $\omega = q^{-2}$.

Later we shall see that the variables q and μ used here are the same as those appearing in (1.7). On the other hand (1.7) can be rewritten in the form

$$\check{R}^{ab}_{cd}(\mu\, \lambda) = q^\rho \frac{(1-q^{-2})^n}{[n]!}\prod_{j=1}^{n}\alpha_{c+j-1}(\lambda)\beta_{d-j+1}(\mu)[d-j+1]|_{n=b-c=d-a} \tag{3.14}$$

where

$$\alpha_{m-1}(\lambda)\beta_m(\lambda)=[\lambda-m+1]$$
$$\rho=\tfrac{1}{2}\lambda\mu+2ab-a\lambda-b\mu-\tfrac{1}{2}n(n+1)+n(b-a)-\tfrac{1}{2}n(\lambda-\mu). \tag{3.15}$$

By using (3.15), equation (3.14) can be recast to

$$\check{R}^{ab}_{cd}(\mu,\lambda)=q^{\rho}\frac{(q-q^{-1})^n}{[n]!}\prod_{j=1}^{n}\frac{[\alpha-j+1]}{\alpha_{d-j}(\mu)}\frac{\alpha_{c+j-1}(\lambda)}{[c+j]}[c+j][\mu-a-j+1]. \tag{3.16}$$

Which is related to (3.12) by

$$\check{R}^{ab}_{cd}(\mu,\lambda)=q^{\frac{1}{2}\lambda\mu}(q^{\frac{1}{2}\mu})^{(b+c)}(q^{\frac{1}{2}\lambda})^{-(a+d)}\frac{H(\mu,a)H(\lambda,b)}{H(\mu,d)H(\lambda,c)}\tilde{G}^{ab}_{cd}(\alpha,\beta;-) \tag{3.17}$$

where

$$H(\mu,a)=\frac{\prod_{j=0}^{a-1}\alpha_{a-j}(\mu)}{[a]!}\qquad \alpha=q^{2\mu}\qquad \text{and } \omega=q^{-2}. \tag{3.18}$$

Obviously the coefficients before $\tilde{G}^{ab}_{cd}(\alpha,\beta;-)$ in (3.17) are nothing but the transformations (3.7) and (3.9). We thus conclude that (3.1) and (1.7) are related to each other by successive coloured symmetry breaking transformations in the straightforward way. The relationship between the parameters is given by $\alpha=q^{2\mu}$ and $\omega=q^{-2}$, consistent with the previous calculations [12].

Acknowledgment

We would like to thank Professor L H Kauffman for valuable discussions. This work is in part supported by NSF of China.

References

[1] Lee H C, Couture M and Schmeing N C 1988 *Preprint* CRNL TP-88-1118r
Lee H C (ed) 1990) *Physics Geometry and Topology* (New York: Plenum) pp 573–82
[2] Ge M L, Liu X F and Sun C P 1991 *Phys. Lett.* **155A** 137
[3] Ge M L, Sun C P and Xue K 1992 Construction of general colored R-matrices for YBE and q-boson realization of quantum algebra $SL_q(2)$ when q is a root of unity *Int. J. Mod. Phys.* A **7** 2763
[4] Sun C P and Ge M L 1991 q-boson realization theory of quantum algebra and its applications to YBE, Nankai Lectures on Mathematical Physics (April 1991)
Ge M L (ed) 1991 *Quantum Group and Quantum Integrable Systems* (Singapore: World Scientific) pp 133–229
[5] Drinfeld V G 1986 *Proc. ICM, Berkely* p 798; 1985 *Sov. Math. Dokl.* **32** 254
[6] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlan A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
Kulish P P and Damashinsky E 1990 *J. Phys. A: Math. Gen.* **23** L415
[7] Akutsu Y and Deguchi T 1991 *Phys. Rev. Lett.* **67** 777
[8] Deguchi T and Akutsu Y 1990 *J. Phys. A: Math. Gen.* **23** 1861
[9] Deguchi T and Akutsu Y 1991 *J. Phys. Soc. Japan* **60** 4051
[10] Akutsu Y and Wadati M 1987 *J. Phys. Soc. Japan* **56** 839
[11] Marakami J 1989 A state model for the multi-variable Alexander polynomial *Preprint* Osaka
[12] Ge M L, Liu G C, Sun C P and Wang Y W 1992 Non-Birman–Wenzl algebraic properties and redundancy of exotic enhanced YB operator for spin model *Preprint* Nankai